The viscous secondary flow ahead of an infinite cylinder in a uniform parallel shear flow

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A simple method is presented in this paper for calculating
(a) the secondary velocities, and
(b) the lateral displacement of total pressure surfaces (i.e. the 'displacement effect')
in the plane of symmetry ahead of an infinitely long cylinder situated normal to a steady, incompressible, slightly viscous shear flow; the cylinder is also perpendicular to the vorticity, which is assumed uniform but small. The method is based on lateral gradients of pressure, these being calculated from the primary flow alone. Profiles of the secondary velocities are obtained at several Reynolds numbers ahead of two specific cylindrical shapes: a circular cylinder, and a flat plate normal to the flow. The displacement effect is derived and, rather surprisingly, is found to be virtually independent of the Reynolds number.

1. Introduction

The inviscid mathematical theory of the secondary flow caused by an infinite cylinder situated perpendicular to a steady, incompressible, weakly sheared parallel flow, and also to the vorticity of that stream, was developed by Lighthill (1956) in a paper titled 'Drift'. He investigated the bending and stretching inflicted upon vortex lines as they are convected around the cylinder by the primary flow along planes perpendicular to the cylinder. Lighthill found a remarkably simple expression for the secondary flow thus induced, which showed that to the order of the approximation involved this flow is entirely parallel to the cylinder, and is independent of the distance along the generators.

When one tries to apply this theory to derive certain results of practical interest, namely, the magnitude of the cross-flow very near the cylinder, and the 'displacement effect', certain shortcomings of the theory become apparent. The 'displacement effect' is the distance by which a curve of pressure measured at holes flush with the upstream face of a cylinder, and plotted against height, would appear shifted—in the direction of decreasing total pressure—from the true total pressure profile of the undisturbed flow. A particular application for this displacement effect arises in determining a non-uniform velocity profile by means of a transverse cylindrical total pressure probe (e.g. Livesey 1956).

The inviscid theory predicts a logarithmically infinite cross-flow velocity on the surface of the cylinder. But actually this velocity reaches a maximum a short distance away, and of course vanishes at the surface itself because of the no-slip condition. It might be argued that the logarithmic singularity is a mild one, and that even a wild guess of the thickness of the boundary layer of the secondary flow could be used to provide a fair approximation of the actual maximum secondary velocity. This turns out to be true. However, there remains the fact that inviscid theory predicts a displacement effect which is likewise infinite, and it is harder to see any means of adjusting this result so as to give a reasonable estimate. In short, a viscous theory from the outset would be desirable, and the present note is a contribution towards that end.

The physical basis of the current work will be that when the shear is assumed small, the fluid moves almost precisely in a two-dimensional manner along planes imagined drawn perpendicular to the cylinder, although in each of these laminae the upstream velocity is one which is appropriate to that particular value of the ordinate. However, as velocities vary slightly between adjoining planes, not only will the fluid be subjected to stresses which determine its essentially two-dimensional motion, but it will also be subjected to a small lateral pressure gradient which causes a certain amount of cross-flow.

The approximation which 'stacks' these two-dimensional (and, in general, viscous) flows one above the other will be referred to in this paper as the *primary flow*. The term *secondary flow* will denote the first-order correction to the former; it is, in fact, the above cross-flow. We shall assume that higher-order corrections will be much smaller, and shall ignore them. The important thing to bear in mind about this approach is that the secondary velocity is considered to arise from fluid elements being convected with the *known* primary velocities across *known* lateral pressure gradients, determined from the primary flow alone. At the same time, they are also subjected to shear stresses arising from their lateral motion, and must obey boundary conditions.

Although this problem is considerably simpler than trying to solve simultaneous Navier–Stokes equations in three dimensions, in its complete form it still remains an ambitious task. In order to further simplify matters, one might now be tempted to suppose that the boundary layer of the secondary flow is considerably thicker than that of the primary flow, and thus to regard the primary flow as inviscid at least around the forepart of the cylinder. Such an assumption would, unfortunately, be mistaken. The thicknesses of these boundary layers are of the same order because the relevant speeds of convection are the same for both. This means that a complete viscous secondary flow solution will inescapably involve the primary boundary layer; and the computation of that alone is known to be very cumbersome.

For this reason calculations have been carried out in this paper only for the flow in the plane of symmetry that extends upstream from the cylinder. Fortunately, we can obtain useful results there, but with much reduced labour. (As a matter of convenience, we consider only cylinders that are symmetrical, and hence non-lifting, though of otherwise fairly arbitrary shape.) In the plane of symmetry, the geometry and the assumed small uniform shear together permit significant simplifications to be made in the equation of motion for the cross-flow component of velocity. As a result, just an ordinary linear differential equation has to be solved to obtain the desired secondary flow. The primary flow in this plane is taken to consist of an outer part calculated from ordinary potential flow and matched near the cylinder to Hiemenz's solution for viscous stagnation point flow. To conclude §2, the above methods are used to compute some secondary velocity profiles for two particular cylinders.

The displacement effect is studied in §3, where it is found, to a good first approximation, that this does not depend on the Reynolds number. It is indicated that an intuitively based adjustment of the inviscid theory to find this quantity would have been considerably mistaken. Next, the somewhat unrealistic nature of the assumption of *unbounded* shear flow is discussed in connexion with the displacement effect which would be observed in practice. Finally, two specific examples are again worked out in detail.

2. Secondary flow velocity

Choose a system of right-handed Cartesian co-ordinates as illustrated in figure 1. The velocity far upstream has the components

$$v_x = U_1(y) = U + Ay, \quad v_y = v_z = 0, \tag{1}$$

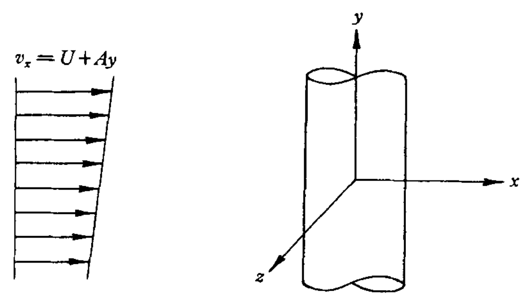

FIGURE 1. Cylinder in shear flow, showing co-ordinate axes.

U and A being constants. It is thus already assumed that the flow is parallel and that the shear can be approximated as uniform. We further require that the latter also be small, or that

$$aA/U_1 \ll 1, \tag{2}$$

where a is a typical dimension of the cylinder. (In the two examples to be treated later, $2a$ = width.) It might also be added, to avoid any possible confusion, that the region of interest does not include $U_1 < 0$.

The first approximation to the actual flow by the primary flow has been mentioned already. It supposes that the actual velocity $\mathbf{v}$ at any point (x, y_0, z) just equals the velocity $\mathbf{v}_1(x,z;y_0)$ that would result from a *uniform* stream of speed $U_1(y_0)$ flowing past the cylinder. Consequently, v_{1x} and v_{1z} must together obey the two-dimensional equations of motion and continuity, and satisfy the boundary conditions $\mathbf{v}_1 = 0$ at the surface, and $\mathbf{v}_1 = (U_1, 0, 0)$ at infinity. However, $v_{1y} = 0$ everywhere. Note that if the Reynolds number R did not vary with the ordinate y, then the following similarity relation would hold exactly:

$$\mathbf{v}_1(x,z;y) = (U+Ay)\,\mathbf{u}(x,z). \tag{3}$$

As it is, R varies slowly, but (3) remains sufficiently exact to be used hereafter.

It becomes evident on closer examination that the primary flow quite accurately depicts the actual motion in the planes perpendicular to the cylinder, but does not do so well as regards the flow parallel to the cylinder. We begin by writing down the equation of motion for v_y:

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v_y. \tag{4}$$

Near the cylinder $\partial p/\partial y$ is generally of the order of $(\partial/\partial y)\,[\frac{1}{2}\rho(U+Ay)^2] = \rho A U_1$. Consequently, v_y is of the order of $\int \frac{1}{\rho v_x}\frac{\partial p}{\partial y}\,dx$ or of $O(aA)$, whereas the primary flow had estimated it as zero. It is interesting that v_y is independent of y to a rough approximation; this can be confirmed by more detailed analysis (e.g. equation (9)). This was also one of Lighthill's conclusions.

On the other hand, if one were to write the dynamical equation for v_x (or for v_z) and in it insert $v_x = v_{1x}$ (or $v_z = v_{1z}$), then a number of terms would at once drop out since they would constitute the two-dimensional equation of motion, leaving only the terms $\nu(\partial^2 v_{1x}/\partial y^2)$ and $v_y(\partial v_{1x}/\partial y)$ (or their analogues). The former is quite negligible because of (3). The latter is of $O(Av_y)$, whereas a typical vanished term such as $v_x(\partial v_x/\partial x)$ would be of $O(U^2/a)$. This suggests that errors in v_x and v_z caused by representing the actual flow by the primary flow alone amount to $UO(aAv_y/U^2) = O(a^2A^2/U)$, or only about (aA/U) times those for v_y.

Of course, a solution must also satisfy the equation of continuity

$$\operatorname{div} \mathbf{v} = 0. \tag{5}$$

But since v_{1x} and v_{1z} already obey its two-dimensional equivalent, and since $v_y = O(aA)$ is not a function of y to the first approximation, there is no difficulty on this point. Thus we are now justified in stating that the *secondary* flow consists entirely of motion directed parallel to the cylinder. Only one velocity component, v_y, remains to be computed, whereas $v_x \doteqdot v_{1x}$ and $v_z \doteqdot v_{1z}$ are already presumed known as functions of position. In addition, the pressure and thus $\partial p/\partial y$ are already determined from the requirements of the primary flow.

It will be of interest to consider briefly the secondary flow which can be calculated from this approach should the fluid be *inviscid*. In that case, (4) would reduce to

$$v_{1x} \frac{\partial v_y}{\partial x} + v_{1z} \frac{\partial v_y}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{6}$$

where it has been reasonably assumed that $v_y(\partial v_y/\partial y)$ is negligible. Representing by u the magnitude of the vector $\mathbf{u}$ which was defined by (3), and noting that Bernoulli's equation,

$$p - p_\infty = \tfrac{1}{2}\rho(U_1^2 - v_1^2), \tag{7}$$

gives

$$\frac{\partial p}{\partial y} = \rho A U_1 (1-u^2), \tag{8}$$

it becomes possible to write this explicit solution of (6):

$$-\frac{v_{yi}}{aA} = h_i(x) = \int_{s\,(\text{at }x)}^{s_0\,(\text{at }x_0)} \frac{1-u^2(s)}{u(s)}\,ds. \tag{9}$$

The integration is to proceed along a streamline of the primary flow, and $ds = -(1/a)(dx^2+dz^2)^{\frac{1}{2}}$. Equation (9) also assumes that $v_y = 0$ at $x = x_0$; the reason why x_0 does not necessarily equal $-\infty$ appears in the last paragraphs of § 3.

It is a simple matter to demonstrate that when $x_0 = -\infty$, (9) agrees exactly with Lighthill's corresponding result (equation (43)), which he obtained from vorticity considerations. The coincidence should cause no surprise, as the methods are, of course, fundamentally related. On the other hand, it does serve as a cross-check to show that the approximations involved are identical. This modest success of the pressure gradient method should, however, not disguise the fact that the vorticity method seems definitely superior for calculating higher-order corrections to the inviscid flow.

After these preliminaries, let us confine our subsequent attention to the viscous problem in that part of the plane of symmetry, $z = 0$, which lies upstream of the cylinder, and where considerable simplifications of (4) are justified. In this plane one omits quite readily the terms $v_y(\partial v_y/\partial y)$ and $v_z(\partial v_y/\partial z)$, the former because $v_y \ll U_1$ and $(\partial v_y/\partial y) = 0$, and the latter because $v_z = 0$ due to symmetry. The term $\nu(\partial^2 v_y/\partial y^2)$ is also thoroughly negligible, again because of $(\partial v_y/\partial y) = 0$. The only omission which calls for lengthier consideration is that of $\nu(\partial^2 v_y/\partial z^2)$. The ratio of this term to the significant viscous term $\nu(\partial^2 v_y/\partial x^2)$ is usually of $O(\delta^2/a^2)$, where δ is the stagnation point boundary layer thickness; thus the approximation

$$\partial^2 v_y/\partial z^2 \ll \partial^2 v_y/\partial x^2 \tag{10}$$

requires an R at least large enough to make $a^2 \gg \delta^2$. However, it should be noted that the word 'usually' presumes that the cylinder is sufficiently blunt-faced. (By way of contrast, a forward-facing wedge, with an only slightly blunted leading edge of radius b, would here require that $b^2 \gg \delta^2$, a more stringent requirement than the previous one.)

Adopting these restrictions, (4) becomes

$$v_{1x}\frac{dv_y}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{d^2 v_y}{dx^2} \tag{11}$$

in the plane of symmetry. Together with the boundary conditions

$$v_y(x = x_c) = v_y(x = x_0) = 0, \tag{12}$$

where the subscript c refers to the cylinder surface, this is a well-set problem.

However, rather than trying to solve (11) outright, it is profitable to split up the solution into a viscous inner part and a matching outer part for which the viscous term in (11), if not wholly disregarded, is treated as only a small perturbation. Since δ/a is so small, $\partial p/\partial y$ can be assumed equal to $\rho A U_1$ throughout the entire viscous solution. Moreover, the primary flow can there be approximated by a viscous stagnation point flow against a flat plate for which the Hiemenz solution is well known (Goldstein 1938). On the other hand, a solution already exists for the outer part in the form of (9); although, in order to be more accurate, $u(s)$ might have to be derived from a potential flow around the combined profile of the cylinder and the displacement thickness of the boundary layer and the wake.

For the inner solution, it is best to replace x by the Hiemenz variable

$$\eta = \frac{1}{a}(u'R)^{\frac{1}{2}}(x_c - x). \tag{13}$$

Here u' is the value of $\partial u/\partial s = -(a/U_1)(\partial v_{1x}/\partial x)$ at $x = x_c$ that would arise from inviscid primary flow; and the Reynolds number is given a definite meaning by $R = aU_1/\nu$. Then v_{1x} is

$$v_{1x} = U_1(u'/R)^{\frac{1}{2}} f_1(\eta), \tag{14}$$

$f_1(\eta)$ being a well-tabulated function (see Goldstein 1938, p. 151); also,

$$f_1 \sim \eta - 0{\cdot}6479$$

for large η. Put

$$v_y = -aAh(\eta) = -\frac{aA}{u'}q(\eta). \tag{15}$$

Consequently, (11) transforms into

$$\frac{d^2q}{d\eta^2} + f_1(\eta)\frac{dq}{d\eta} = -1. \tag{16}$$

Putting $f_0(\eta) = \int_0^\eta f_1(\eta')\,d\eta'$, the solution of (16) is

$$\begin{aligned} q(\eta) &= -\int_0^\eta e^{-f_0(\zeta)}\int_0^\zeta e^{+f_0(\zeta')}\,d\zeta'\,d\zeta + K\int_0^\eta e^{-f_0(\zeta)}\,d\zeta \\ &= -q_{\mathrm{I}}(\eta) + Kq_{\mathrm{II}}(\eta). \end{aligned} \tag{17}$$

Here K is an arbitrary constant equal to $\partial q(0)/\partial\eta$; another constant has already been disposed of by the requirement $q(0) = 0$.

The integrals $q_{\mathrm{I}}(\eta)$ and $q_{\mathrm{II}}(\eta)$ were calculated up to $\eta = 6$ numerically. These results are summarized in table 1. Knowing that for large η

$$f_0 \sim \tfrac{1}{2}(\eta - 0{\cdot}6479)^2 + const., \tag{18}$$

η	$q_{\mathrm{I}}(\eta)$	$q_{\mathrm{II}}(\eta)$	η	$q_{\mathrm{I}}(\eta)$	$q_{\mathrm{II}}(\eta)$
0·0	0·000	0·000	3·0	2·261	1·733
0·2	0·020	0·200	3·2	2·369	1·741
0·4	0·080	0·399	3·4	2·465	1·747
0·6	0·178	0·594	3·6	2·551	1·750
0·8	0·312	0·782	3·8	2·629	1·751
1·0	0·476	0·959	4·0	2·700	1·752
1·2	0·663	1·120	4·2	2·765	1·753
1·4	0·867	1·262	4·4	2·826	1·753
1·6	1·077	1·384	4·6	2·883	1·753
1·8	1·286	1·483	4·8	2·936	1·753
2·0	1·487	1·562	5·0	2·986	1·753
2·2	1·674	1·623	5·2	3·034	1·753
2·4	1·846	1·667	5·4	3·079	1·753
2·6	2·001	1·698	5·6	3·123	1·753
2·8	2·139	1·719	5·8	3·164	1·753
3·0	2·261	1·733	6·0	3·204	1·753

TABLE 1

we can calculate that

$$q(\xi) \sim 1{\cdot}753K - 1{\cdot}545 - \ln \xi + \tfrac{1}{2}\xi^{-2} + \tfrac{3}{4}\xi^{-4}\ldots, \tag{19}$$

where ξ denotes $(\eta - 0{\cdot}6479)$, and where the values of the first two constants have been supplied by the numerical integration. It has been pointed out by Mr E. J. Watson that the integral $q_{\text{II}}(\eta)$ is also encountered in the study of the boundary layer on a yawed cylinder.

As regards the outer solution, one might not wish to omit the viscous term in (11) altogether. However, we would still use (9) to compute v_y initially to an inviscid approximation, namely v_{yi}. The first-order viscous correction to this, say $v_{y'}$, would then be calculated from the following adaptation of (11):

$$v_{1x}\frac{dv_{y'}}{dx} = \nu\frac{d^2v_{yi}}{dx^2}, \tag{20}$$

or, more to the point, from the solution of (20) given as

$$-\frac{v_{y'}(x)}{aA} = \frac{1}{R}\left(\frac{1}{u^2(x)} - \ln u(x)\right). \tag{21}$$

By way of illustration, the foregoing analysis is now applied to two specific examples: (i) a circular cylinder (denoted by CC) of radius a, and (ii) a flat plate normal to the stream (FP) of width $2a$. In both cases it is assumed that the primary flow outside the boundary layer is adequately described by the ordinary potential flow around these cylinders. (This constitutes probably the largest single source of error here.) Thus,

$$\text{CC:}\quad u(s) = 1 - s^{-2}; \qquad \text{FP:}\quad u(s) = s(1+s^2)^{-\frac{1}{2}} \tag{22}$$

(Milne-Thomson 1955). In order to facilitate the matching of the inner and outer solutions, it is desirable that $u(s) = 0$ at a distance from the surface equal to the displacement thickness

$$\delta = 0{\cdot}6479a(u'R)^{-\frac{1}{2}}; \tag{23}$$

hence the somewhat artificial definition

$$s = -\frac{1}{a}(x+\delta). \tag{24}$$

Here $x = 0$ at the centres of both cylinders.

Assuming that $v_y = 0$ at $x = -\infty$ for the present, we now insert (22) into (9) and integrate to obtain the approximate inviscid outer solutions

$$\left.\begin{aligned} \text{CC:}\quad -v_{yi}/aA &= \frac{1}{s} + \tfrac{1}{2}\ln\frac{s+1}{s-1}, \\ \text{FP:}\quad -v_{yi}/aA &= \ln\frac{s}{(1+s^2)^{\frac{1}{2}} - 1}. \end{aligned}\right\} \tag{25}$$

For small values of ξ, (25) can be expanded as

$$\left.\begin{aligned} \text{CC:}\quad q(\xi) &= 2 + \tfrac{1}{2}\ln(8R) - \ln\xi - \frac{3\xi}{8R} + \frac{15}{32}\frac{\xi^2}{R}\cdots, \\ \text{FP:}\quad q(\xi) &= \ln 2 + \tfrac{1}{2}\ln R - \ln\xi + \frac{\xi^2}{4R}\cdots. \end{aligned}\right\} \tag{26}$$

A viscous correction to be added to (26) is obtained from (21). When ξ is small, it is found for either of the cylinders that this correction is approximately

$$\Delta q(\xi) = \xi^{-2}. \tag{27}$$

Selecting the value of ξ at which the inner and outer solutions are to be matched is somewhat a matter of choice. Here we have decided to equate (19) with the sum of (26) and (27) at $\xi = 5$ for $R = 10^3$, 10^4, 10^5 and 10^6. The resulting secondary velocity profiles are displayed in figure 2.

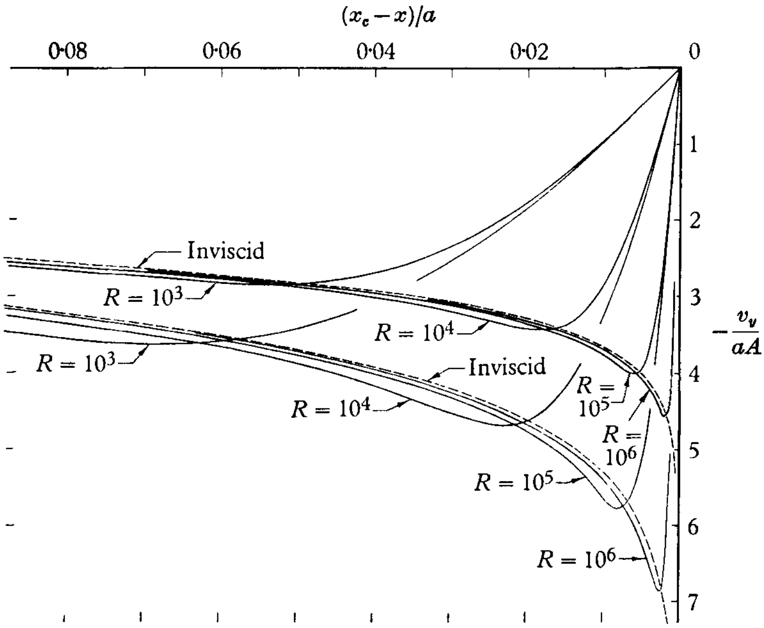

FIGURE 2. Profiles of the viscous secondary flow in the plane of symmetry in front of a circular cylinder (upper curves) and a flat plate (lower curves).

3. Displacement effect

Let us return once more to the equation of motion, which we now write in the following well-known form:

$$\nabla(\tfrac{1}{2}\rho v^2) + \rho \mathbf{w} \times \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v}, \tag{28}$$

where $\mathbf{w} = \text{curl } \mathbf{v}$. Consequently, if p_0 denotes the total pressure $(\frac{1}{2}\rho v^2 + p)$, then

$$\frac{\partial}{\partial x} p_0 + \rho(\mathbf{w} \times \mathbf{v})_x = \mu \nabla^2 \mathbf{v}_x. \tag{29}$$

In the plane of symmetry, $(\mathbf{w} \times \mathbf{v})_x = -w_z v_y$, since $w_y = v_z = 0$. We write approximately, as before,

$$v_x = (U + Ay)\, u(x), \quad \text{and} \quad v_y = -aA\, h(x). \tag{30}$$

Hence
$$w_z = -\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} = -A\left\{u(x) + a \frac{\partial}{\partial x} h(x)\right\}. \tag{31}$$

Insert these into (29), and simplify $\nabla^2 v_x$ to $\partial^2 v_x/\partial x^2$, partly by arguments similar to those which accompanied (10) and partly because $\partial^2 v_x/\partial y^2 = 0$ from (3). Then integrate the result with respect to x. This leaves

$$\left[p_0 - \mu\frac{\partial v_x}{\partial x} - \frac{\rho a^2 A^2}{2} h^2(x')\right]_{x_0}^{x_c} = \rho a A^2 \int_{x_0}^{x_c} h(x')\, u(x')\, dx'. \tag{32}$$

Since $\partial v_x(x_0)/\partial x = \partial v_x(x_c)/\partial x = 0$, and $h(x_0) = h(x_c) = 0$, we observe that the total pressure at the surface exceeds the total pressure at the same ordinate far upstream, at $x = x_0$, by the amount

$$\Delta p_0 = \rho a A^2 \int_{x_0}^{x_c} h(x')\, u(x')\, dx'. \tag{33}$$

It is known that the total pressure gradient, $\partial p_0/\partial y$, is $\rho A U_1$, and therefore we conclude that the displacement of the total pressure towards the region of lower pressure is given by

$$\frac{\Delta y}{a} = \frac{aA}{U_1}\int_{s_c}^{s_0} h(s')\, u(s')\, ds'. \tag{34}$$

Here the dimensionless s_0 and s_c correspond to x_0 and x_c, respectively.

It happens that the contribution to this expression from a region extending only a small distance Δx from the surface is roughly of the order of

$$(aA/U_1)\,(\Delta x/a)^2.$$

Since the entire $\Delta y/a = O(aA/U_1)$, (34) can to good accuracy be calculated using the $u(s)$ and $h(s)$ that are given by the inviscid outer solution, omitting the tiny contribution from the inner solution altogether.

Thus, the remarkable thing about the displacement effect as described by (34) is its virtual independence from the Reynolds number. This is hardly what one would have anticipated from inviscid considerations. The displacement of streamlines, and therefore that of constant total pressure loci, from s_0 to some s nearer the cylinder would be given by inviscid theory instead as

$$\frac{\Delta y(s)}{a} = \frac{aA}{U_1}\int_{s}^{s_0} \frac{h(s')}{u(s')}\, ds'. \tag{35}$$

Although (35) is such that† $\lim_{s\to s_c} \Delta y(s) = \infty$, there is no reason to doubt its correctness even for the present case, provided only that s remains sufficiently large to exclude the boundary layer. However, one would have thought that by inserting a lower limit $s = s_c + \delta = s_c + 0{\cdot}6479 a(u'R)^{-\frac{1}{2}}$, from (23), one might obtain a rough estimate for $\Delta y/a$, and that this would consequently strongly depend on R. But (34) indicates that this is not so.

The surprising behaviour of the surfaces of constant total pressure occurs entirely in the boundary layer. Here one deduces from (32) that their displacement is in fact larger at any distance $O(\delta)$ from the cylinder than at the surface itself. The decrease in the displacement near the surface as (34) implies, must necessarily be of an amount sufficient just to cancel out any effects of the

† This remains the case even if the h and u obtained from viscous theory are inserted in (35).

Reynolds number. (To make these remarks seem more plausible, recall that in a Poiseuille flow too, a given total pressure surface lies further downstream anywhere in the channel than at the walls.)

A difficulty remains, however, about what to do with the upper limit of (34), s_0, which has so far remained unspecified. The reason why it cannot be chosen equal to ∞ here is that Δy turns out to behave as $\ln s_0$ at large s_0 for any cylinder whose effect at large distances upstream can be approximated by that of a line dipole, i.e. any closed cylinder in inviscid flow. For an unclosed cylinder, such as that formed by a line source, or a closed cylinder with a wake, this behaviour is evident in the secondary velocity itself. In that case Δy grows linearly with s_0 as $s_0 \to \infty$! This handicap seems to be an unavoidable, but not necessarily an erroneous, feature of the theory for an unbounded shear flow containing an infinite cylinder.

It is almost certain, however, that the concept of *unbounded* shear flow is unrealistic at large distances. Not only are actual shear flows usually generated not very far upstream, but also, more importantly, they are in practice either bounded by walls or confined to layers. It seems physically unlikely that these lateral constraints would permit appreciable secondary flow to extend to a distance upstream of more than several times the typical lateral dimension d.

The idea of a cut-off distance x_0 is therefore attractive. For the purposes of calculating the displacement effect, the secondary velocity would be presumed to begin only when the flow reaches $x = x_0 = O(d)$. Lighthill (1957) considered a similar cut-off in connexion with the displacement effect for a source in a bounded shear flow, and in fact derived an integral expression for estimating this distance. It should be possible in principle to extend his methods to our cylinders, by regarding them as composed of suitable arrangements of line sources and sinks. But bearing in mind the complexity of the methods, and that this cut-off distance will generally vary from one streamline to another (even in the simple case of a uniform shear flow in a channel), we will not attempt to calculate x_0 here. It will suffice to remember that this distance depends only on the flow profile and boundaries, and neither on the size nor the shape of the cylinder.

The displacement effects for the circular cylinder and the flat plate are worked out below as functions of the above-mentioned cut-off distance. If we could assume that $u(s)$ is accurately described by the potential flow around these cylinders, we would take $u(s)$ from (22), and $h(s)$ as implied by (25), integrate (34) and so find

$$\left.\begin{aligned} \text{CC:}\quad \Delta y/a &= (aA/U_1)\left(2\ln s_0 + \frac{4}{s_0} - \frac{3}{2s_0^2} + \ldots - 2{\cdot}886\right), \\ \text{FP:}\quad \Delta y/a &= (aA/U_1)\left(\ln s_0 + \frac{1}{s_0} - \frac{1}{6s_0^3} + \ldots - 0{\cdot}693\right). \end{aligned}\right\} \tag{36}$$

To be realistic, though, these bodies have to be considered together with their wakes. Let us represent a wake by a line source through the origin. Consequently the expression for $u(s)$ takes the form

$$u(s) \doteqdot 1 - (C_D/2\pi)\, s^{-1} - (\text{Potential flow terms of } O(s^{-2})). \tag{37}$$

Here C_D denotes the drag coefficient defined as the drag per unit length divided by $\rho U_1^2 a$, and the fact has been used that the displacement thickness of a wake tends to $\frac{1}{2}C_D$ times the cylinder width. Applying equation (37) to (9), we see that the wake must thus contribute to the non-dimensional secondary flow velocity an amount

$$h_{\text{wake}} \doteqdot (C_D/\pi)\ln(s_0/s) \tag{38}$$

(as long as s remains somewhat larger than unity). From (34) it then follows that this is responsible for displacing the total pressure surfaces by approximately an additional amount

$$\Delta y/a \doteqdot (aA/U_1)(C_D/\pi)(s_0 - \ln s_0), \tag{39}$$

which must be added to (36) for the combined effect of the cylinder and wake. We observe that when the drag coefficient is of order unity as here, then s_0 need not assume particularly large values before the contribution of the wake to the displacement effect becomes the dominant one. This is the case despite the fact that the fraction of the secondary velocity observed near the cylinder which has been caused by the wake may typically be only, say, a quarter of the total.

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